Nonlinear Stefan Problems in One-Space Dimension : An Approach by the Theory of Subdifferential Operators (Applied Analysis for Partial Differential Equations in Applied Sciences)

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Nonlinear Stefan Problems in One-Space Dimension
(An Approach by the Theory of Subdifferential Operators)

Nobuyuki KENMOCHI

Department of Mathematics, Faculty of Education, Chiba University, JAPAN

Introduction.

In the physical processes we find many phenomena with change of state, such as melting of ice, recrystallization of metals, evaporation, condensation and flow in porous media. Stefan problems in mathematics represent such physical models and have been studied by many authors (e.g., [3, 4, 6 - 20, 25 - 28, 30, 32, 33, 35] and their references).

The present paper is devoted to the study of Stefan problems for nonlinear parabolic differential equations of the forms

$$(*) \qquad u_t - \beta(u)_{xx} = f, \qquad\qquad (**) \qquad u_t - (|u_x|^{p-2}u_x)_x = f$$

in one-space dimension, where β: $R \to R$ is a given function with $\beta(0) = 0$ which is strictly increasing and bi-Lipschitz continuous, and p is a number satisfying $2 \leq p < \infty$. Equations (*) and (**) are special cases of the following general form

$$(***) \qquad u_t - (|\beta(u)_x|^{p-2}\beta(u)_x)_x = f,$$

so we shall deal with equation (***) in what follows.

Given a non-negative number ℓ_o, functions g on $[0, T]$ (T is a fixed positive number), u_o on $[0, \ell_o]$ and f on $[0, T] \times [0, \infty)$, our problem (one phase Stefan problem) is to find a non-negative function $x = \ell(t)$ on $[0, T]$ and a function $u = u(t, x)$ on $[0, T] \times [0, \infty)$ such that

(E) $u_t - (|\beta(u)_x|^{p-2}\beta(u)_x)_x = f$ for $0 < t < T$, $\ell(t) > 0$, $0 < x < \ell(t)$

subject to

(C1) $\ell(0) = \ell_o$ and if $\ell_o > 0$, then $u(0, x) = u_o(x)$ for $0 < x < \ell_o$,

(C2) $\begin{cases} |\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+) = g(t) & \text{for } 0 < t < T, \\ \beta(u)(t, \ell(t)) = 0 & \text{for } 0 < t < T, \end{cases}$

(C3) $\dfrac{d\ell(t)}{dt} = - |\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \ell(t)-)$ for $0 < t < T$,

where $\beta(u)_x(t, x+)$ (resp. $\beta(u)_x(t, x-)$) stands for the right (resp. left) hand partial derivative of $\beta(u)(t, x)$ at x with respect to x. The unknown boundary $x = \ell(t)$ is called the free boundary.

To this kind of problems there are such approaches as listed below:

(a) difference method (cf. [32, 35]),

(b) approach by the theory of nonlinear semigroups (cf. [3, 7, 25]),

(c) reduction to variational inequalities (cf. [8, 9, 11, 16 - 18, 28]),

(d) reduction to nonlinear integral equations (cf. [12, 15, 26, 32]),

etc. In this paper we employ (d) with (c). A class of nonlinear Stefan problems was treated earlier by Kyner [26], in which he established an existence and uniqueness theorem by employing (d) and by making use of a strong maximum principle of Nirenberg [29] for parabolic equations with variable coefficients. But his method is not directly available to our case, in particular to the case that $p > 2$ or β is not smooth. Our approach to problem {(E), (C1) - (C3)}, which is different from that of Kyner in some points of view, is based upon recent results on the existence, uniqueness and stability of solutions to nonlinear evolution equations involving subdifferential operators of time-dependent convex functions on Hilbert spaces (cf. [1, 2, 21 - 23, 31, 34, 36]).

Notations. For a (real) Banach space V we denote by $|\cdot|_V$ the norm in V, by V^* its dual and by $(\cdot\,,\cdot)_V$ the duality pairing between V^* and V; in case V is a Hilbert space and is identified with its dual space, we mean by $(\cdot\,,\cdot)_V$ the inner product in V.

By an operator A from a Banach space V into another Banach space W we mean that to each v in V, A assigns a subset Av of W, namely A is a multi-valued mapping from V into W; in particular, if Av consists of at most one element of W for every v in V, then A is called singlevalued. For an operator $A: V \to W$ the set $D(A) = \{v \in V;\ Av \neq \phi\}$ is called the domain.

Let ϕ be a lower semi-continuous convex function on a Hilbert space H with values in $(-\infty, \infty]$ such that $\phi \not\equiv \infty$ on H. Then the set $D(\phi) = \{z \in H;\ \phi(z) < \infty\}$ is called the effective domain of ϕ, and the subdifferential $\partial\phi$ of ϕ is an operator from H into itself defined as follows: $z^* \in \partial\phi(z)$ if and only if $z \in D(\phi)$, $z^* \in H$ and

$$(z^*, y - z)_H \leqq \phi(y) - \phi(z), \qquad \forall\, y \in H.$$

For fundamental properties of $\partial\phi$ we refer to a book of Brézis [5].

1. Formulation as a quasi-variational problem.

Let $2 \leqq p < \infty$ and $0 < T < \infty$ be numbers which are fixed, and set for simplicity

$$H = L^2(0, \infty), \qquad X = W^{1,p}(0, \infty).$$

Let $\beta: R \to R$ be a function with $\beta(0) = 0$ and assume that β is strictly increasing and bi-Lipschitz continuous on R, i.e.,

$$\frac{1}{c_\beta}|r - r_1|^2 \leqq (\beta(r) - \beta(r_1))(r - r_1) \leqq c_\beta|r - r_1|^2, \quad \forall\, r, r_1 \in R$$

with a positive constant c_β.

Given a non-negative continuous function ℓ: $[0, T] \to R$ and a continuous function g: $[0, T] \to R$, we define for each t in $[0, T]$

$$K_\ell(t) = \{z \in X;\ z(x) = 0,\ \ \forall\ x \geq \ell(t)\}$$

and

$$\phi^t_{\ell,g}(z) = \begin{cases} \frac{1}{p}\int_0^\infty |z_x|^p dx + g(t)z(0) & \text{if } z \in K_\ell(t), \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

Clearly $\phi^t_{\ell,g}$ is a lower semi-continuous convex function on H with $D(\phi^t_{\ell,g}) = K_\ell(t)$. We then consider the nonlinear evolution equation

$$u'(t) + \partial\phi^t_{\ell,g}(Bu(t)) \ni f(t) \qquad \text{for } 0 < t < T, \tag{1.2}$$

where the unknown u is an H-valued function on $[0, T]$, $u'(t) = (d/dt)u(t)$ and B is the singlevalued operator from $H = D(B)$ into itself defined by

$$[Bz](x) = \beta(z(x)) \qquad \text{for } z \in H \text{ and } x \in [0, \infty).$$

By our assumption we see that B is Lipschitz continuous on H with c_β as a Lipschitz constant as well as B^{-1} with $1/c_\beta$ as a Lipschitz constant.

Definition 1.1. Let ℓ, g be as above, u_o be in H and f in $L^2(0, T; H)$. Then we mean by $VP(\ell, g, u_o, f)$ the Cauchy problem for (1.2) to find a function u in $C([0, T]; H)$ such that

(A1) $u \in W^{1.2}(0, T; H)$ and $u(0) = u_o$;

(A2) the function $t \to \phi^t_{\ell,g}(Bu(t))$ is bounded on $[0, T]$;

(A3) $u'(t) + \partial\phi^t_{\ell,g}(Bu(t)) \ni f(t)$ for a.e. t in $[0, T]$.

Such a function u is called a (strong) solution to $VP(\ell, g, u_o, f)$.

Proposition 1.1. Let ℓ, g, u_o, f be as in Definition 1.1. Then a solution u to $VP(\ell, g, u_o, f)$ is able to be characterized by the following system:

$$(1.3)\qquad \begin{cases} u \in W^{1,2}(0, T; H) \text{ with } u(0) = u_o, \\ \beta(u) \in L^{\infty}(0, T; X), \\ \beta(u)(t, \cdot) \in K_{\ell}(t) \text{ (hence } \beta(u)(t, \ell(t)) = 0) \text{ for all } t \in [0, T], \end{cases}$$

$$(1.4)\qquad u_t(t, \cdot) - (|\beta(u)_x(t, \cdot)|^{p-2}\beta(u)_x(t, \cdot))_x = f(t, \cdot)$$
$$\text{in the distribution sense on } (0, \ell(t)) \text{ for a.e. } t \in I_o,$$

$$(1.5)\qquad |\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+) = g(t) \quad \text{for a.e. } t \in I_o,$$

where $I_o = \{t \in [0, T];\ \ell(t) > 0\}$. Moreover, $\beta(u)_x(t, \ell(t)-)$ exists for a.e. $t \in I_o$.

Proof. Obviously (1.3) follows from (A1) and (A2). As is easily seen, (A3) can be written in the following equivalent form:

$$(1.6)\qquad \begin{cases} (u'(t) - f(t), z)_H + \int_0^{\infty} |\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x) z_x(x)dx \\ \qquad + g(t)z(0) = 0, \quad \forall\ z \in K_{\ell}(t), \quad \text{for a.e. } t \in [0, T]. \end{cases}$$

We see from (1.6) that (1.4) holds and hence $(|\beta(u)_x(t, \cdot)|^{p-2}\beta(u)_x(t, \cdot))_x = u_t(t, \cdot) - f(t, \cdot) \in L^2(0, \ell(t))$ for a.e. $t \in I_o$. This implies that $|\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x)$ is an absolutely continuous function of x on $(0, \ell(t))$ and $\beta(u)_x(t, 0+)$ exists for a.e. $t \in I_o$ as well as $\beta(u)_x(t, \ell(t)-)$, so that by integration by parts we obtain (1.5) from (1.6). Similarly we can show the converse. Q.E.D.

Now we are going to give a quasi-variational formulation associated with our Stefan problem {(E), (C1) - (C3)}.

Definition 1.2. Let ℓ_o be a non-negative number, u_o be in H, g in $C([0, T])$ and f in $L^2(0, T; H)$. Then we mean by QVP(ℓ_o, g, u_o, f) to find a couple $\{\ell, u\}$ such that

(B1) $\ell \in W^{1,2}(0, T)$ and $\ell \geq 0$ on $[0, T]$;

(B2) u is a solution to VP(ℓ, g, u_o, f);

(B3) $$\ell(t) = \ell_o - \int_0^t g(r)dr + \int_0^{\ell_o} u_o(x)dx + \int_0^t\int_0^{\ell(r)} f(r, x)dxdr - \int_0^\infty u(t, x)dx \qquad \text{for all } t \in [0, T].$$

Proposition 1.2. Let ℓ_o, g, u_o, f be as in Definition 1.2 and $\{\ell, u\}$ be a solution to QVP(ℓ_o, g, u_o, f). Assume that g is non-positive and u_o, f are non-negative. Then (1.3), (1.4) and the following (1.5)', (1.7) hold:

(1.5)' $$|\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+) = g(t) \qquad \text{for a.e. } t \in [0, T],$$

(1.7) $$\frac{d\ell(t)}{dt} = -|\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \ell(t)-) \quad \text{for a.e. } t \in [0, T].$$

Proof. By Proposition 1.1 we have (1.3), (1.4), (1.5) and $\beta(u)_x(t,\ell(t)-)$ exists for a.e. $t \in I_o$ (= $\{t \in [0, T];\ \ell(t) > 0\}$). From (B3) with (1.5) it follows that

$$\begin{aligned}\frac{d\ell(t)}{dt} &= -g(t) + \int_0^{\ell(t)} f(t, x)dx - \int_0^\infty u_t(t, x)dx \\ &= -g(t) - \int_0^{\ell(t)} (|\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x))_x dx \\ &= -|\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \ell(t)-)\end{aligned}$$

for a.e. $t \in I_o$. Also, if $t \in [0, T] - I_o$, then $u(t, x) = 0$ for all $x \geqq 0$ and (B3) implies $g(t) = 0$. Therefore

$$|\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+) = 0 = g(t) \qquad \text{for all } t \in (0, T] - I_o$$

and

$$\frac{d\ell(t)}{dt} = 0 = -|\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \beta(t)-) \quad \text{for a.e. } t \in [0, T] - I_o.$$

Thus (1.5)' and (1.7) are satisfied. Q.E.D.

By the above proposition we see that $QVP(\ell_0, g, u_0, f)$ is a quasi-variational problem associated with $\{(E), (C1) - (C3)\}$.

Our results on $QVP(\ell_0, g, u_0, f)$ are stated as follows.

Theorem 1.1. Let $\ell_0 \geqq 0$, $u_0 \in H$ be non-negative, $g \in C([0, T])$ be non-positive and $f \in L^2(0, T; H)$ be non-negative. Then we have:

(a) If $\{\ell, u\}$ is a solution to $QVP(\ell_0, g, u_0, f)$, then u is non-negative and ℓ is non-decreasing in t.

(b) In addition suppose that $u_0 \in X$, $u_0 = 0$ on $[\ell_0, \infty)$ and $g \in W^{1,1}(0, T)$. Then $QVP(\ell_0, g, u_0, f)$ has at least one solution.

The proof of this theorem will be given in sections 2 and 5. In order to demonstrate the existence of a solution to $QVP(\ell_0, g, u_0, f)$ we shall introduce a mapping P from a certain compact convex subset S of $C([0, T])$ into itself defined as follows:

$$(1.8)\qquad \begin{aligned}[P\ell](t) = \ell_0 - \int_0^t g(r)dr + \int_0^{\ell_0} u_0(x)dx + \int_0^t\int_0^{\ell(r)} f(r, x)dxdr \\ - \int_0^\infty u^\ell(t, x)dx \qquad \text{for each } \ell \in S \text{ and } t \in [0, T],\end{aligned}$$

where u^ℓ is a solution to $VP(\ell, g, u_0, f)$. We shall show that there is an element ℓ of S satisfying $P\ell = \ell$ by a fixed point theorem and that the couple $\{\ell, u^\ell\}$ is a solution to $QVP(\ell_0, g, u_0, f)$.

The problem of uniqueness for a solution to $QVP(\ell_0, g, u_0, f)$ remains open, but in the special case that $p = 2$ and $f \equiv 0$ we shall show

Theorem 1.2. If $p = 2$, then $QVP(\ell_0, g, u_0, f)$ has at most one solution for $\ell_0 \geqq 0$, $g \in C([0, T])$ non-positive, $u_0 \in X$ non-negative with $u_0 = 0$ on $[\ell_0, \infty)$ and $f \equiv 0$.

2. A comparison theorem for $VP(\ell, g, u_o, f)$.

We show the following comparison theorem for solutions to $VP(\ell, g, u_o, f)$.

Theorem 2.1. Let ℓ be a non-negative function in $C([0, T])$, g, $\bar{g}$ be in $C([0, T])$ with $g \leqq \bar{g}$ on $[0, T]$, u_o, $\bar{u}_o$ in H and f, $\bar{f}$ in $L^2(0, T; H)$. Let u and $\bar{u}$ be solutions to $VP(\ell, g, u_o, f)$ and $VP(\ell, \bar{g}, \bar{u}_o, \bar{f})$, respectively. Then we have:

$$
\begin{aligned}
|(\bar{u}(t) - u(t))^+|_{L^1(0, L)} \leqq\ & |(\bar{u}(s) - u(s))^+|_{L^1(0, L)} \\
& + \int_s^t |(\bar{f}(r) - f(r))^+|_{L^1(0, L)} dr
\end{aligned} \tag{2.1}
$$

for any $0 \leqq s \leqq t \leqq T$ and any positive number $L \geqq |\ell|_{C([0, T])}$, where $(\ \cdot\ )^+$ stands for the positive part of $(\ \cdot\ )$.

Proof. Take a sequence $\{\sigma_n\}$ of smooth functions on R with non-negative bounded derivatives σ_n' such that $-1 \leqq \sigma_n \leqq 1$ on R, $\sigma_n(0) = 0$ and

$$
\sigma_n(r) \longrightarrow \sigma_o(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0 \\ -1 & \text{if } r < 0 \end{cases} \qquad (\text{as } n \to \infty)
$$

for each $r \in R$. Since $\sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+) \in K_\ell(t)$ for all t in $[0, T]$, we have by (1.6) in the proof of Proposition 1.1

$$
\begin{aligned}
&(u'(t) - f(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+))_H \\
&+ \int_0^\infty |\beta(u)_x(t, x)|^{p-2} \beta(u)_x(t, x) \frac{\partial}{\partial x} \sigma_n([\beta(\bar{u})(t, x) - \beta(u)(t, x)]^+) dx \\
&+ g(t)\sigma_n([\beta(\bar{u})(t, 0) - \beta(u)(t, 0)]^+) = 0
\end{aligned}
$$

and

$$
(\bar{u}'(t) - \bar{f}(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+))_H
$$

$$+ \int_0^\infty |\beta(\bar{u})_x(t, x)|^{p-2}\beta(\bar{u})_x(t, x)\frac{\partial}{\partial x}\sigma_n([\beta(\bar{u})(t, x) - \beta(u)_x(t, x)]^+)dx$$

$$+ \bar{g}(t)\sigma_n([\beta(\bar{u})(t, 0) - \beta(u)(t, 0)]^+) = 0$$

for a.e. $t \in [0, T]$, from which we infer that

$$(\bar{u}'(t) - u(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+))_H$$

$$\leqq (\bar{f}(t) - f(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+))_H$$

$$- \int_0^\infty \{|\beta(\bar{u})_x(t, x)|^{p-2}\beta(\bar{u})_x(t, x) - |\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x)\} \times$$

$$\times \frac{\partial}{\partial x}\sigma_n([\beta(\bar{u})(t, x) - \beta(u)(t, x)]^+)dx$$

$$- (\bar{g}(t) - g(t))\sigma_n([\beta(\bar{u})(t, 0) - \beta(u)(t, 0)]^+)$$

$$= (\bar{f}(t) - f(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+))_H$$

$$- \int_{E(t)} \{|\beta(\bar{u})_x(t, x)|^{p-2}\beta(\bar{u})_x(t, x) - |\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x)\} \times$$

$$\times (\beta(\bar{u})_x(t, x) - \beta(u)_x(t, x))\sigma_n'([\beta(\bar{u})(t, x) - \beta(u)(t, x)]^+)dx$$

$$- (\bar{g}(t) - g(t))\sigma_n([\beta(\bar{u})(t, 0) - \beta(u)(t, 0)]^+)$$

$$\leqq (\bar{f}(t) - f(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+))_H$$

for a.e. $t \in [0, T]$, where $E(t) = \{x; \beta(\bar{u})(t, x) > \beta(u)(t, x)\}$. Letting $n \to \infty$, we get for a.e. t in $[0, T]$

(2.2) $$(\bar{u}'(t) - u'(t), \sigma_o([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+))_H \leqq |(\bar{f}(t)-f(t))^+|_{L^1(0,L)}.$$

Since $\sigma_o([\beta(\bar{u})(t, x) - \beta(u)(t, x)]^+) = \sigma_o([\bar{u}(t, x) - u(t, x)]^+)$ and

$$(\bar{u}'(t) - u'(t), \sigma_o([\bar{u}(t, \cdot) - u(t, \cdot)]^+))_H = \frac{d}{dt}|(\bar{u}(t) - u(t))^+|_{L^1(0, L)},$$

it follows from (2.2) that

$$\frac{d}{dt}|(\bar{u}(t) - u(t))^+|_{L^1(0, L)} \leqq |(\bar{f}(t) - f(t))^+|_{L^1(0, L)}$$

for a.e. $t \in [0, T]$. Integrating this inequality, we get (2.1). Q.E.D.

Remark 2.1. The technic adopted above is found in Bénilan [3] and Damlamian [8].

The following corollaries are immediate consequences of Theorem 2.1.

Corollary 1. Let ℓ, g, u_o and f be as in Theorem 2.1. Then $VP(\ell, g, u_o, f)$ has at most one solution.

Corollary 2. Let ℓ, g, u_o and f be as in Theorem 2.1 and further assume that g is non-positive and u_o, f are non-negative. Then a solution to $VP(\ell, g, u_o, f)$ is non-negative.

Proof of (a) of Theorem 1.1: Let $\{\ell, u\}$ be any solution to $QVP(\ell_o, g, u_o, f)$. Then, according to Corollary 2 to Theorem 2.1, u is non-negative as well as $\beta(u)$. Therefore

$$\beta(u)_x(t, \ell(t)-) \leqq 0 \qquad \text{for a.e. } t \in [0, T]$$

and by (1.7) of Proposition 1.2

$$\frac{d\ell(t)}{dt} \geqq 0 \qquad \text{for a.e. } t \in [0, T],$$

which shows that ℓ is non-decreasing in t. Q.E.D.

3. Some lemmas.

Throughout this section we fix $\ell \in C([0, T])$ and $g \in W^{1,1}(0, T)$, and assume that ℓ is non-decreasing and non-negative. For simplicity by ϕ^t we denote the convex function $\phi^t_{\ell,g}$ given by (1.1).

First of all we show a lemma on the t-dependence of ϕ^t.

Lemma 3.1. With $C = (|g|_{C([0, T])} + 1)^{p'}\ell(T) + 1$, $p' = p/(p - 1)$, we have

$$(3.1) \qquad \phi^s(z) \geqq -C, \qquad \forall\, s \in [0, T], \quad \forall\, z \in K_\ell(s),$$

$$(3.2) \qquad \phi^t(z) - \phi^s(z) \leqq C|g(t) - g(s)|(\phi^s(z) + C),$$
$$\forall\, s, t \in [0, T] \text{ with } s \leqq t, \ \forall\, z \in K_\ell(s).$$

Proof. Let $0 \leqq s \leqq T$ and $z \in K_\ell(s)$. Then for any $\delta > 0$ we observe that

$$|z(0)| \leqq \int_0^{\ell(s)} |z_x|dx \leqq \int_0^{\ell(s)} \{\frac{\delta|z_x|^p}{p} + \frac{\delta^{1-p'}}{p'}\}dx$$

$$= \frac{\delta}{p}\int_0^{\ell(s)} |z_x|^p dx + \frac{\delta^{1-p'}\ell(s)}{p'}$$

$$= \delta\phi^s(z) - \delta g(s)z(0) + \frac{\delta^{1-p'}\ell(s)}{p'},$$

so that by taking $\delta = (|g|_{C([0, T])} + 1)^{-1}$ we have

$$\phi^s(z) = \frac{1}{p}\int_0^{\ell(s)} |z_x|^p dx + g(s)z(0)$$

$$\geqq \frac{1}{p}(1 - \delta|g|_{C([0, T])})\int_0^{\ell(s)} |z_x|^p dx - \frac{|g|_{C([0, T])}\delta^{1-p'}\ell(T)}{p'}$$

$$\geqq -C$$

and for any $t > s$

$$\phi^t(z) - \phi^s(z) = (g(t) - g(s))z(0)$$

$$\leqq \frac{\delta}{1 - \delta|g|_{C([0, T])}}|g(t) - g(s)|(\phi^s(z) + \frac{\delta^{-p'}\ell(T)}{p'})$$

$$\leqq C|g(t) - g(s)|(\phi^s(z) + C).$$

Thus (3.1) and (3.2) hold. Q.E.D.

Next, we consider the regularization ϕ^t_λ, $0 < \lambda \leqq 1$, of ϕ^t as is defined by

$$\phi^t_\lambda(z) = \inf_{y \in H} \{\frac{1}{2\lambda}|z - y|^2_H + \phi^t(y)\}, \qquad z \in H.$$

In what follows, without proof we shall use some of well-known properties of ϕ_λ^t such as

(i) ϕ_λ^t is finite, continuous and convex on H,

(ii) the subdifferential $\partial\phi_\lambda^t$: $H = D(\partial\phi_\lambda^t) \to H$ is singlevalued and Lipschitz continuous with $1/\lambda$ as a Lipschitz constant,

(iii) $J_\lambda^t = (I + \lambda\partial\phi^t)^{-1}$ is contractive on H and $\partial\phi_\lambda^t = (I - J_\lambda^t)/\lambda$,

(iv) $J_\mu^t z = J_\lambda^t(\frac{\lambda}{\mu}z + (1 - \frac{\lambda}{\mu})J_\mu^t z)$ for μ, $\lambda > 0$ and $z \in H$,

(v) $\phi_\lambda^t(z) = |z - J_\lambda^t z|_H^2/(2\lambda) + \phi^t(J_\lambda^t z)$ for $z \in H$,

etc., for which we can refer to a book of Brézis [5].

Lemma 3.2. $\phi_\lambda^t(z) \geqq - C$ for any $t \in [0, T]$, $\lambda \in (0, 1]$ and $z \in H$, where C is the same constant as in Lemma 3.1.

Proof. By (3.1) of Lemma 3.1,

$$\phi_\lambda^t(z) = \frac{1}{2\lambda}|z - J_\lambda^t z|_H^2 + \phi^t(J_\lambda^t z) \geqq \phi^t(J_\lambda^t z) \geqq - C. \qquad \text{Q.E.D.}$$

Lemma 3.3. Let C be as in Lemma 3.1. Then we have:

(i) $|J_\lambda^t z|_H \leqq |z|_H + 2\sqrt{2C}$, $\forall\, t \in [0, T]$, $\forall\, \lambda \in (0, 1]$, $\forall\, z \in H$;

(ii) $|\partial\phi_\lambda^t(z)|_H \leqq \frac{2}{\lambda}(|z|_H + \sqrt{2C})$, $\forall\, t \in [0, T]$, $\forall\, \lambda \in (0, 1]$, $\forall\, z \in H$;

(iii) $- C \leqq \phi_\lambda^t(z) \leqq \frac{2}{\lambda}(|z|_H + \sqrt{2C})|z|_H$, $|\phi_\lambda^t(z) - \phi_\lambda^t(z_1)| \leqq \frac{2}{\lambda}(|z|_H + |z_1|_H + 2\sqrt{2C})|z - z_1|_H$, $\forall\, t \in [0, T]$, $\forall\, \lambda \in (0, 1]$, $\forall\, z, z_1 \in H$.

Proof. Using Lemma 3.2, we observe

$$0 \geqq \phi_1^t(0) = \frac{1}{2}|J_1^t 0|_H^2 + \phi^t(J_1^t 0) \geqq \frac{1}{2}|J_1^t 0|_H^2 - C,$$

so

$$|J_1^t 0|_H \leqq \sqrt{2C}, \qquad \forall\, t \in [0, T].$$

Making use of the relation $J_1^t z = J_\lambda^t(\lambda z + (1 - \lambda)J_1^t z)$, $z \in H$, we see that

$$|J_\lambda^t 0|_H \leqq |J_1^t 0|_H + (1-\lambda)|J_1^t 0|_H \leqq 2|J_1^t 0|_H$$

and hence

(3.3) $\qquad |J_\lambda^t 0|_H \leqq 2\sqrt{2C}, \quad \forall\, t \in [0, T], \quad \forall\, \lambda \in (0, 1].$

Since J_λ^t is contractive on H, we have by (3.3)

$$|J_\lambda^t z|_H \leqq |J_\lambda^t z - J_\lambda^t 0|_H + |J_\lambda^t 0|_H \leqq |z|_H + |J_\lambda^t 0|_H \leqq |z|_H + 2\sqrt{2C}$$

for any $t \in [0, T]$, $\lambda \in (0, 1]$ and $z \in H$, which shows (i). Next from (i) it follows that

$$|\partial\phi_\lambda^t(z)|_H = \frac{1}{\lambda}|z - J_\lambda^t z|_H \leqq \frac{2}{\lambda}(|z|_H + \sqrt{2C})$$

for any $t \in [0, T]$, $\lambda \in (0, 1]$ and $z \in H$. Hence we have (ii). To show (iii) we note that

$$\phi_\lambda^t(z) - \phi_\lambda^t(z_1) \leqq (\partial\phi_\lambda^t(z), z - z_1)_H \leqq |\partial\phi_\lambda^t(z)|_H |z - z_1|_H,$$

$$\phi_\lambda^t(z) - \phi_\lambda^t(z_1) \geqq (\partial\phi_\lambda^t(z_1), z - z_1)_H \geqq -|\partial\phi_\lambda^t(z_1)|_H |z - z_1|_H$$

and

$$\phi_\lambda^t(z) \leqq |\partial\phi_\lambda^t(z)|_H |z|_H$$

for any $t \in [0, T]$, $\lambda \in (0, 1]$ and $z, z_1 \in H$. From these inequalities, (ii) and Lemma 3.2 we derive that

$$\begin{aligned} |\phi_\lambda^t(z) - \phi_\lambda^t(z_1)| &\leqq \{|\partial\phi_\lambda^t(z)|_H + |\partial\phi_\lambda^t(z_1)|_H\}|z - z_1|_H \\ &\leqq \frac{2}{\lambda}(|z|_H + |z_1|_H + 2\sqrt{2C})|z - z_1|_H \end{aligned}$$

and

$$-C \leqq \phi_\lambda^t(z) \leqq \frac{2}{\lambda}(|z|_H + \sqrt{2C})|z|_H$$

for any $t \in [0, T]$, $\lambda \in (0, 1]$ and $z, z_1 \in H$. Thus we get (iii). Q.E.D.

The following lemma is due to Attouch-Bénilan-Damlamian-Picard [1] (or Picard [31]).

Lemma 3.4. Let $v \in W^{1,1}(0, T; H)$ and $0 < \lambda \leqq 1$. Then $\phi_\lambda^t(v(t))$ is differentiable at a.e. $t \in [0, T]$ and its derivative is integrable on $[0, T]$. Moreover, with the same constant C as in Lemma 3.1 we have

$$(3.4) \qquad \phi_\lambda^t(v(t)) - \phi_\lambda^s(v(s)) \leqq \int_s^t \frac{d}{dr}\phi_\lambda^r(v(r))dr \qquad \text{for any } 0 \leqq s \leqq t \leqq T,$$

$$(3.5) \qquad \frac{d}{dt}\phi_\lambda^t(v(t)) - (\partial\phi_\lambda^t(v(t)), v'(t))_H \leqq C|g'(t)|(\phi_\lambda^t(v(t)) + C)$$

for a.e. $t \in [0, T]$.

Proof. Let $z \in H$. Then by Lemma 3.1 and (iii) of Lemma 3.3 we observe that for $0 \leqq s < t \leqq T$

$$(3.6) \qquad \begin{aligned} \phi_\lambda^t(z) - \phi_\lambda^s(z) &\leqq \phi^t(J_\lambda^s z) - \phi^s(J_\lambda^s z) \\ &\leqq C|g(t) - g(s)|(\phi_\lambda^s(z) + C) \\ &\leqq c(z)\int_s^t |g'(r)|dr \end{aligned}$$

with

$$c(z) = C\{\tfrac{2}{\lambda}(|z|_H + \sqrt{2C})|z|_H + C\}.$$

From (3.6) we see that

$$t \longrightarrow \phi_\lambda^t(z) - c(z)\int_0^t |g'(r)|dr$$

is non-increasing on $[0, T]$, so that $\phi_\lambda^t(z)$ is differentiable at a.e. t in $[0, T]$, its derivative is integrable on $[0, T]$ and

$$(3.7) \qquad \phi_\lambda^t(z) - \phi_\lambda^s(z) \leqq \int_s^t \frac{d}{dr}\phi_\lambda^r(z)dr \qquad \text{for any } 0 \leqq s < t \leqq T.$$

Moreover,

$$(3.8) \qquad \frac{d}{dr}\phi_\lambda^r(z) \leqq C|g'(r)|(\phi_\lambda^r(z) + C) \qquad \text{for a.e. } r \in [0, T].$$

Next, let $v \in W^{1,1}(0, T; H)$. Using (3.7) and (3.8), we have

$$
(3.9)\quad
\begin{aligned}
&\phi_\lambda^t(v(t)) - \phi_\lambda^s(v(s)) - (\partial\phi_\lambda^t(v(t)),\ v(t) - v(s))_H \\
&\leqq \phi_\lambda^t(v(s)) - \phi_\lambda^s(v(s)) \\
&\leqq \int_s^t \frac{d}{dr}\phi_\lambda^r(v(s))dr \\
&\leqq C\int_s^t |g'(r)|(\phi_\lambda^r(v(s)) + C)dr
\end{aligned}
$$

for any $0 \leqq s < t \leqq T$, from which (3.4) follows just as (3.7). Besides, dividing (3.9) by $t - s$ and letting $s \uparrow t$, we obtain (3.5) with the help of (iii) of Lemma 3.3. Q.E.D.

Finally we show

Lemma 3.5. For each $\lambda \in (0, 1]$, the operator $v \to \partial\phi_\lambda^{(\cdot)}(v(\cdot))$ is Lipschitz continuous on $L^2(0, T; H)$ with $1/\lambda$ as a Lipschitz constant.

Proof. As was seen in the proof of Lemma 3.4, $t \to \phi_\lambda^t(v(t))$ is measurable on $[0, T]$ for each v in $W^{1,1}(0, T; H)$ and hence for each v in $L^2(0, T; H)$. Now we consider a function Φ^λ on $L^2(0, T; H)$ which is defined by

$$\Phi^\lambda(v) = \int_0^T \phi_\lambda^t(v(t))dt, \qquad \forall\ v \in L^2(0, T; H).$$

Obviously Φ^λ is finite, continuous and convex on $L^2(0, T; H)$ (cf. Lemma 3.3). It is also easy to see that the subdifferential $\partial\Phi^\lambda$: $L^2(0, T; H) \to L^2(0, T; H)$ is a singlevalued operator with $D(\partial\Phi^\lambda) = L^2(0, T; H)$ and

$$\partial\Phi^\lambda(v) = \partial\phi_\lambda^{(\cdot)}(v(\cdot)), \quad \forall\ v \in L^2(0, T; H),$$

so that $\partial\phi_\lambda^t(v(t))$ is (strongly) measurable in $t \in [0, T]$ for every v in $L^2(0, T; H)$. Since $\partial\phi_\lambda^t$ is Lipschitz continuous on H with $1/\lambda$ as a Lipschitz constant for all $t \in [0, T]$, so is $\partial\Phi^\lambda$ on $L^2(0, T; H)$ with the same Lipschitz constant. Q.E.D.

4. Existence and stability of solutions to VP(ℓ, g, u_o, f).

We first establish an existence theorem for VP(ℓ, g, u_o, f).

Theorem 4.1. Let ℓ be a non-negative and non-decreasing function in $C([0, T])$. Let u_o be in $K_\ell(0)$, g in $W^{1,1}(0, T)$ and f in $L^2(0, T; H)$. Then VP(ℓ, g, u_o, f) admits a (unique) solution.

Let ℓ, g, u_o and f be in Theorem 4.1. Then in order to construct a solution to VP(ℓ, g, u_o, f) we consider an approximate problem of the following type:

$$u_\lambda'(t) + \partial(\phi^t_{\ell,g})_\lambda(Bu_\lambda(t)) \ni f(t), \tag{4.1}$$

$$u_\lambda(0) = u_o,$$

where $0 < \lambda \leqq 1$ and $(\phi^t_{\ell,g})_\lambda$ is the regularization of $\phi^t_{\ell,g}$. On account of Lemma 3.5, the operator $v \to \partial\phi_\lambda^{(\cdot)}(Bv(\cdot))$ is Lipschitz continuous on $L^2(0, T; H)$ with c_β/λ as a Lipschitz constant. Therefore for each λ in $(0, 1]$, there is a unique function u_λ in $W^{1,2}(0, T; H)$ satisfying (4.1) a.e. on $[0, T]$ and $u_\lambda(0) = u_o$. We want to show that for a certain sequence $\{\lambda_n\}$ with $\lambda_n \downarrow 0$ (as $n \to \infty$) u_{λ_n} converges to a function in $C([0, T]; H)$ and this limit is a solution to VP(ℓ, g, u_o, f).

For simplicity we denote $\phi^t_{\ell,g}$ by ϕ^t again. To get estimations for $\{u_\lambda;\ 0 < \lambda \leqq 1\}$ which are independent of λ, multiply (4.1) by $(d/dt)Bu_\lambda(t)$. We then have

$$(u_\lambda'(t), \frac{d}{dt}Bu_\lambda(t))_H + (\partial\phi^t_\lambda(Bu_\lambda(t)), \frac{d}{dt}Bu_\lambda(t))_H = (f(t), \frac{d}{dt}Bu_\lambda(t))_H$$

for a.e. $t \in [0, T]$. We note here that

$$(u_\lambda'(t), \frac{d}{dt}Bu_\lambda(t))_H \geqq \frac{1}{c_\beta}|\frac{d}{dt}Bu_\lambda(t)|_H^2,$$

$$(f(t), \frac{d}{dt}Bu_\lambda(t))_H \leqq c_\beta^3|f(t)|_H^2 + \frac{1}{4c_\beta}|\frac{d}{dt}B_\lambda u(t)|_H^2$$

and by Lemma 3.4

$$(\partial\phi_\lambda^t(Bu_\lambda(t)), \frac{d}{dt}Bu_\lambda(t))_H \geqq \frac{d}{dt}\phi_\lambda^t(Bu_\lambda(t)) - C|g'(t)|(\phi_\lambda^t(Bu_\lambda(t)) + C)$$

for a.e. $t \in [0, T]$, where C is the same constant as in Lemma 3.1. Therefore,

$$(4.2)\qquad \frac{3}{4c_\beta}|\frac{d}{dt}Bu_\lambda(t)|_H^2 + \frac{d}{dt}\phi_\lambda^t(Bu_\lambda(t)) \leqq k_1(t)\phi_\lambda^t(Bu_\lambda(t)) + k_2(t)$$

for a.e. $t \in [0, T]$, where

$$k_1(t) = C|g'(t)|, \qquad k_2(t) = C^2|g'(t)| + c_\beta^3|f(t)|_H^2.$$

Applying Granwall's inequality to (4.2), we have

$$(4.3)\qquad \phi_\lambda^t(Bu_\lambda(t)) \leqq \{\phi^0(Bu_o) + \int_0^T k_2(r)dr\}\exp(\int_0^T k_1(r)dr) \equiv C'$$

for all $t \in [0, T]$ and

$$(4.4)\qquad \frac{3}{4c_\beta}|\frac{d}{dt}Bu_\lambda|^2_{L^2(0, T; H)} \leqq C + \phi^0(Bu_o) + \int_0^T\{C'k_1(r) + k_2(r)\}dr.$$

From (4.3) and (4.4) we obtain the following lemma.

Lemma 4.1. There is a positive constant M such that

$$|Bu_\lambda(\cdot)|_{W^{1,2}(0, T; H)} \leqq M \text{ (hence } |u_\lambda|_{W^{1,2}(0, T; H)} \leqq c_\beta M), \ \forall \lambda \in (0, 1]$$

and

$$-C \leqq \phi_\lambda^t(Bu_\lambda(t)) \leqq M, \qquad \forall t \in [0, T], \quad \forall \lambda \in (0, 1],$$

where C is as in Lemma 3.1.

This lemma guarantees that $\{Bu_\lambda(\cdot);\ 0 < \lambda \leqq 1\}$ (hence $\{u_\lambda;\ 0 < \lambda \leqq 1\}$) is relatively compact in $C([0, T]; H)$, because the natural injection from X into H is compact. Now we can choose a sequence $\{\lambda_n\}$ with $\lambda_n \downarrow 0$ such that

$$u_n = u_{\lambda_n} \to u, \qquad Bu_n(\cdot) \to Bu(\cdot) \qquad \text{in } C([0, T]; H)$$

and

$$u_n' \to u' \qquad \text{weakly in } L^2(0, T; H).$$

Evidently $u \in W^{1,2}(0, T; H)$ and $u(0) = u_o$. Putting $u_n^*(t) = \partial\phi_{\lambda_n}^t(Bu_n(t))$, we see that $u_n^* = f - u_n' \to f - u'$ weakly in $L^2(0, T; H)$. Also, since

$$u_n^*(t) = \frac{1}{\lambda_n}(Bu_n(t) - J_{\lambda_n}^t Bu_n(t)) \in \partial\phi^t(J_{\lambda_n}^t Bu_n(t)),$$

we get

$$J_{\lambda_n}^{(\cdot)}Bu_n(\cdot) \to Bu(\cdot) \qquad \text{in } L^2(0, T; H),$$

so the demi-continuity of the operator $v \to \partial\phi^{(\cdot)}(v(\cdot))$ in $L^2(0, T; H)$ gives

$$f(t) - u'(t) \in \partial\phi^t(Bu(t)) \qquad \text{for a.e. } t \in [0, T].$$

Moreover

$$\phi^t(Bu(t)) \leqq \liminf_{n\to\infty} \phi^t(J_{\lambda_n}^t Bu_n(t)) \leqq \liminf_{n\to\infty} \phi_{\lambda_n}^t(Bu_n(t)) \leqq M$$

for all t in $[0, T]$. Thus the function u is a solution to $VP(\ell, g, u_o, f)$ and Theorem 4.1 was completely proved.

Next, we show a stability theorem for $VP(\ell, g, u_o, f)$.

Theorem 4.2. Let ℓ_o and L be numbers such that $0 \leqq \ell_o \leqq L$ and denote by Λ the set $\{\ell \in C([0, T]);\ \ell$ is non-decreasing in t, $\ell(0) = \ell_o$ and $\ell(T) \leqq L\}$. Let $g \in W^{1,1}(0, T)$, $u_o \in X$ with $u_o = 0$ on $[\ell_o, \infty)$ and $f \in L^2(0, T; H)$. Then there is a positive constant K such that

$$|u^\ell|_{W^{1,2}(0, T; H)} \leqq K, \quad |\phi_{\ell,g}^t(Bu^\ell(t))| \leqq K, \quad \forall\, t \in [0, T],$$

for every $\ell \in \Lambda$, where u^ℓ is a unique solution to $VP(\ell, g, u_o, f)$.

Proof. As is easily checked, the constants M in Lemma 4.1 and C in Lemma 3.1 can be taken so as to be bounded, as long as ℓ varies in the set Λ. From this fact the conclusion of the theorem follows. Q.E.D.

This stability theorem plays an important role in solving $QVP(\ell_o, g, u_o, f)$.

5. Proof of (b) of Theorem 1.1.

Throughout this section, assume that $\ell_o \geqq 0$, $u_o \in X$ is non-negative with $u_o = 0$ on $[\ell_o, \infty)$, $g \in W^{1,1}(0, T)$ is non-positive and $f \in L^2(0, T; H)$ is non-negative, and let Λ be the set $\{\ell \in C([0, T]);\ \ell$ is non-decreasing in t, $\ell(0) = \ell_o$ and $\ell(T) \leqq L\}$ with L satisfying

$$L > \ell_o + \int_0^{\ell_o} u_o(x)dx - \int_0^T g(r)dr + \sqrt{LT}|f|_{L^2(0, T; H)}. \tag{5.1}$$

We now consider the operator P on Λ which is defined by (1.8). Concerning this operator P we have

Lemma 5.1. $P(\Lambda) \subset \Lambda \cap W^{1,2}(0, T)$.

Proof. Let $\ell \in \Lambda$. Then $[P\ell](0) = \ell_o$ and $P\ell \in W^{1,2}(0, T)$, since a unique solution u^ℓ to $VP(\ell, g, u_o, f)$ belongs to $W^{1,2}(0, T; H)$. By Proposition 1.1 we have

$$\begin{aligned}\frac{d}{dt}[P\ell](t) &= -g(t) + \int_0^{\ell(t)} f(t, x)dx - \int_0^{\ell(t)} u_t(t, x)dx \\ &= -g(t) - \int_0^{\ell(t)} (|\beta(u^\ell)_x(t, x)|^{p-2}\beta(u^\ell)_x(t, x))_x dx \\ &= -|\beta(u^\ell)_x(t, \ell(t)-)|^{p-2}\beta(u^\ell)_x(t, \ell(t)-)\end{aligned}$$

for a.e. $t \in I_o$ $(= \{t \in [0, T];\ \ell(t) > 0\})$. Also u^ℓ is non-negative by Corollary 2 to Theorem 2.1 as well as $\beta(u^\ell)$. Hence

$$\beta(u^\ell)_x(t, \ell(t)-) \leqq 0 \qquad \text{for a.e. } t \in I_o,$$

from which it follows that $(d/dt)[P\ell](t) \geqq 0$ for a.e. $t \in I_o$. For a.e. t in $[0, T] - I_o$ we have

$$\frac{d}{dt}[P\ell](t) = -g(t) \geqq 0,$$

because $u^\ell(t, x) = 0$ for all $x \geqq 0$ if $t \in [0, T] - I_o$. Therefore $P\ell$ is non-decreasing. By (5.1), $[P\ell](T) \leqq L$. Thus $P\ell \in \Lambda$. Q.E.D.

Lemma 5.2. P is continuous on Λ with respect to the topology of $C([0, T])$.

Proof. Suppose that $\ell_n \in \Lambda$ and $\ell_n \to \ell$ in $C([0, T])$, and denote by u_n and u the solutions to $VP(\ell_n, g, u_o, f)$ and $VP(\ell, g, u_o, f)$, respectively. Then, on account of Theorem 4.2, there is a constant K such that

(5.2) $$|u_n|_{W^{1,2}(0, T; H)} \leqq K, \quad |\phi^t_{\ell_n,g}(Bu_n(t))| \leqq K, \ \forall n, \ \forall t \in [0, T].$$

We note here that for each n the following holds:

(5.3) $$\begin{cases} \int_0^T (u_n'(t) - f(t), Bu_n(t) - w(t))_H dt \leqq \Phi(w) - \Phi(Bu_n), \\ \qquad \forall\ w \in L^p(0, T; X) \text{ with } w(t) \in K_{\ell_n}(t) \text{ for a.e. } t \in [0, T], \end{cases}$$

where

$$\Phi(w) = \frac{1}{p}\int_0^T\int_0^\infty |w_x(t, x)|^p dxdt + \int_0^T g(t)w(t, 0)dt.$$

By (5.2), $\{u_n\}$ is relatively compact in $C([0, T]; H)$. We want to show that $u_n \to u$ in $C([0, T]; H)$. For this purpose, let $\{u_{n_k}\}$ be any subsequence of $\{u_n\}$ such that $u_{n_k} \to \bar{u}$ (hence $Bu_{n_k} \to B\bar{u}$) in $C([0, T]; H)$. Then we have

$$Bu_{n_k}(t) \to B\bar{u}(t) \quad \text{weakly in } X \text{ for each } t \in [0, T],$$
$$Bu_{n_k} \to B\bar{u} \quad \text{weakly in } L^2(0, T; X)$$
$$u_{n_k}' \to \bar{u}' \quad \text{weakly in } L^2(0, T; H)$$

and by the way

$$\phi^t_{\ell,g}(B\bar{u}(t)) \leqq K, \quad \forall\ t \in [0, T],$$

$$\bar{u} \in W^{1,2}(0, T; H), \quad \bar{u}(0) = u_o,$$

(5.4) $$\liminf_{k\to\infty} \Phi(Bu_{n_k}) \geqq \Phi(B\bar{u}).$$

Now denote by Z the set

$$\{v \in L^p(0, T; X);\ v(t) \in K_\ell(t) \text{ for a.e. } t \in [0, T]\}.$$

Let v be any function in Z and ε be any positive number. Putting $v_\varepsilon(t, x) = v(t, x + \varepsilon)$, we see that $v_\varepsilon(t) \in K_{\ell_n}(t)$ for a.e. $t \in [0, T]$ and for all n sufficiently large. Hence, taking $n = n_k$ with $w = v_\varepsilon$ and letting $k \to \infty$ in (5.3), we obtain by (5.4)

$$\int_0^T (\bar{u}'(t) - f(t),\ B\bar{u}(t) - v_\varepsilon(t))_H dt \leqq \Phi(v_\varepsilon) - \Phi(B\bar{u}).$$

Furthermore, since $v_\varepsilon \to v$ in $L^p(0, T; X)$ and $\Phi(v_\varepsilon) \to \Phi(v)$ as $\varepsilon \downarrow 0$,

$$\int_0^T (\bar{u}'(t) - f(t),\ B\bar{u}(t) - v(t))_H dt \leqq \Phi(v) - \Phi(B\bar{u}).$$

This inequality holds for every v in Z, which is equivalent to

$$f(t) - \bar{u}'(t) \in \partial\phi^t_{\ell,g}(B\bar{u}(t)) \qquad \text{for a.e. } t \in [0, T]$$

(cf. Kenmochi [21; Proposition 1.1]). Thus $\bar{u}$ is a solution to VP(ℓ, g, u_o, f). By the uniqueness of solution we have $\bar{u} = u$. Therefore it must be true that $u_n \to u$ in C([0, T]; H), so that $P\ell_n \to P\ell$ in C([0, T]). Q.E.D.

Proof of (b) of Theorem 1.1: Consider the following subset S of Λ:

$$S = \left\{ \ell \in \Lambda;\ \begin{array}{l} |\ell(t) - \ell(s)| \leqq |t - s||g|_{C([0, T])} \\ \quad + \sqrt{|t - s|}L|f|_{L^2(0, T; H)} + \sqrt{|t - s|}K \\ \quad \text{for all } s, t \in [0, T] \end{array} \right\},$$

where K is the same constant as in Theorem 4.2. Obviously S is compact and convex in C([0, T]) and $P(\Lambda) \subset S$. Taking Lemmas 4.1 and 4.2 into account, we see that P is continuous on S with respect to the topology of C([0, T]) and $P(S) \subset S$. Hence, by a well-known fixed-point theorem there is $\ell \in S$ such that $P\ell = \ell$ and it is easy to see that the couple $\{\ell, u^\ell\}$, u^ℓ being a unique solution to $VP(\ell, g, u_0, f)$, is a solution to $QVP(\ell_0, g, u_0, f)$.

Q.E.D.

6. A uniqueness theorem in a special case.

Throughout this section we assume that $p = 2$, $\ell_0 \geqq 0$, $g \in C([0, T])$ is non-positive and $u_0 \in X$ $(= W^{1,2}(0, \infty))$ is non-negative with $u_0 = 0$ on $[\ell_0, \infty)$.

Let $\{\ell, u\}$ be an arbitrary solution to $QVP(\ell_0, g, u_0, 0)$. Then we know the following facts (cf. Proposition 1.2 and Theorem 1.1):

(6.1) ℓ is non-decreasing with $\ell(0) = \ell_0$ and u is non-negative;

(6.2) $u_t(t, \cdot) - \beta(u)_{xx}(t, \cdot) = 0$ a.e. on $[0, \ell(t)]$ for a.e. $t \in [0, T]$;

(6.3) $\begin{cases} u(0, x) = u_0(x) \text{ for } 0 \leqq x \leqq \ell_0, \ u(t, x) = 0 \text{ for } x \geqq \ell(t) \text{ and} \\ \beta(u)_x(t, 0+) = g(t) \text{ for a.e. } t \in [0, T]; \end{cases}$

(6.4) $\dfrac{d\ell(t)}{dt} = -\beta(u)_x(t, \ell(t)-)$ for a.e. $t \in [0, T]$.

We define

$$v(t, x) = \int_0^t \beta(u)(r, x)dr \quad \text{for } x \geqq 0, \ 0 \leqq t \leqq T$$

and note that

$$v_t(t, x) = \beta(u)(t, x) \geqq 0, \qquad v_x(t, x) = \int_0^t \beta(u)_x(r, x)dr.$$

Now, let η be any function in X. Then we have by (6.1) - (6.4)

$$\int_0^\infty v_x(t, \cdot)\eta_x dx$$

$$= \int_0^t \int_0^{\ell(r)} \beta(u)_x(r, \cdot)\eta_x dxdr$$

$$= \int_0^t \{- \int_0^{\ell(r)} \beta(u)_{xx}(r,\cdot)\eta dx - \beta(u)_x(r,0+)\eta(0) + \beta(u)_x(r,\ell(r)-)\eta(\ell(r))\} dr$$

$$= - \int_0^t\int_0^{\ell(r)} u_t(r, \cdot)\eta dx dr - (\int_0^t g dr)\eta(0) - \int_0^t \frac{d\ell(r)}{dr}\eta(\ell(r)) dr$$

$$= - \int_0^\infty u(t, \cdot)\eta dx + \int_0^\infty u_o \eta dx - (\int_0^t g dr)\eta(0) - \int_0^{\ell(t)} \eta dx,$$

from which we get the following lemma.

Lemma 6.1. Let $\{\ell, u\}$ be a solution to QVP(ℓ_o, g, u_o, 0) and v be as above. Also let ρ be the inverse of β. Then

$$I(t; v, \eta) \equiv \int_0^\infty \rho(v_t)(t,x)(v_t(t,x) - \eta(x))dx + \int_0^\infty v_x(t,x)(v_{xt}(t,x) - \eta_x(x))dx$$

$$- \int_0^\infty u_o(x)(v_t(t,x) - \eta(x))dx + (\int_0^t g(r)dr)(v_t(t,0) - \eta(0))$$

$$+ \int_0^\infty (v_t(t, x) - \eta(x))dx \leqq 0$$

for all $t \in [0, T]$ and all $\eta \in Y = \{\eta \in X;\ \eta$ is non-negative and $\eta(x) = 0$ for all sufficiently large $x\}$.

Proof. We set for simplicity

$$J(t; \eta) = \int_0^\infty \rho(v_t)(t, \cdot)\eta dx + \int_0^\infty v_x(t, \cdot)\eta_x dx - \int_0^\infty u_o \eta dx$$

$$+ (\int_0^t g dr)\eta(0) + \int_0^\infty \eta dx \qquad \text{for } \eta \in Y.$$

As was seen above, $J(t; \eta) \geqq 0$ for all $\eta \in Y$ and $J(t; v_t(t, \cdot)) = 0$. Therefore $I(t; v, \eta) = J(t; v_t(t, \cdot)) - J(t; \eta) \leqq 0$. Q.E.D.

Proof of Theorem 1.2: Let $\{\ell, u\}$ and $\{\bar{\ell}, \bar{u}\}$ be two solutions to QVP(ℓ_o, g, u_o, 0). Then with the same notations as above it follows from

Lemma 6.1 that

$$0 \geqq I(t;\, v,\, \bar{v}_t(t,\,\cdot)) + I(t;\, \bar{v},\, v_t(t,\,\cdot))$$

$$= \int_0^\infty \{\rho(v_t)(t,\, x) - \rho(\bar{v}_t)(t,\, x)\}(v_t(t,\, x) - \bar{v}_t(t,\, x))dx$$

$$+ \int_0^\infty (v_x(t,\, x) - \bar{v}_x(t,\, x))(v_{xt}(t,\, x) - \bar{v}_{xt}(t,\, x))dx$$

$$\geqq \frac{1}{2}\frac{d}{dt}|v_x(t,\,\cdot) - \bar{v}_x(t,\,\cdot)|_H^2$$

for a.e. $t \in [0,\, T]$. This gives

$$|v_x(t,\,\cdot) - \bar{v}_x(t,\,\cdot)|_H \leqq |v_x(0,\,\cdot) - \bar{v}_x(0,\,\cdot)|_H = 0$$

for all $t \in [0,\, T]$, so $v_t = \bar{v}_t$, i.e., $u = \bar{u}$ as well as $\ell = \bar{\ell}$. Q.E.D.

Remark 6.1. The technic adopted above is found in Duvaut [11].

7. Some remarks.

A. Let $\ell_0 > 0$ and g, u_0, f be as in (b) of Theorem 1.1 and let $\{\ell,\, u\}$ be a solution to QVP(ℓ_0, g, u_0, f). Further assume that $g \in W^{1,2}(0,\, T)$. Then $\ell \in W^{1,2}(0,\, T)$ and u is a unique solution to VP(ℓ, g, u_0, f). It is not difficult to verify that the family $\{\phi^t_{\ell,g};\, 0 \leqq t \leqq T\}$ satisfies the following:

$$(7.1)\quad \begin{cases} \text{for each } s,\, t \in [0,\, T] \text{ and } z \in K_\ell(s) \text{ there is } z_1 \in K_\ell(t) \text{ such} \\ \text{that} \\ |z_1 - z|_H \leqq C_0\{|g(t) - g(s)| + |\ell(t) - \ell(s)|\}(|\phi^s_{\ell,g}(z)|^{1/2}+1), \\ |\phi^t_{\ell,g}(z_1)-\phi^s_{\ell,g}(z)| \leqq C_0\{|g(t)-g(s)|+|\ell(t)-\ell(s)|\}(|\phi^s_{\ell,g}(z)|+1), \\ \text{where } C_0 \text{ is a positive constant independent of } s,\, t \text{ and } z. \end{cases}$$

In fact, if we take for z given in $K_\ell(s)$

$$z_1(x) = z(\frac{\ell(s)}{\ell(t)}x), \quad 0 \leqq x < \infty,$$

then we obtain inequalities in (7.1) with a positive constant C_o independent of s, t and z. Under (7.1) it can be shown (cf. Kenmochi [24]) that the function $t \to \phi^t_{\ell,g}(Bu(t))$ is absolutely continuous on [0, T]. This implies that $t \to |\beta(u)(t, \cdot)|_X$ is continuous on [0, T], so that $\beta(u) \in C([0, T]; X)$.

B. Let $\ell_o > 0$, $g \leqq 0$ be in $W^{1,2}(0, T)$ and u_o, f be as in (b) of Theorem 1.1, and let h be in $L^2(0, T)$. Then by QVP(ℓ_o, g, u_o, f, h) we mean the problem to find a couple $\{\ell, u\}$ satisfying (B1), (B2) of Definition 1.2 and the following (B3)' instead of (B3):

$$\text{(B3)'} \quad \ell(t) = \ell_o - \int_0^t g(r)dr + \int_0^t h(r)dr + \int_0^{\ell_o} u_o(x)dx$$
$$+ \int_0^t\int_0^{\ell(r)} f(r, x)dxdr - \int_0^\infty u(t, x)dx, \quad \forall\, t \in [0, T].$$

This integral equation (B3)' is corresponding to Stefan condition of the following type:

$$\frac{d\ell(t)}{dt} = -|\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \ell(t)-) + h(t) \quad \text{for } 0 < t < T.$$

In this case we should notice that the free boundary $x = \ell(t)$ is not necessarily non-decreasing in t. However the same approach is possible to QVP(ℓ_o, g, u_o, f, h).

C. Finally we consider the problem to find a couple $\{\ell, u\}$ satisfying

$$\text{(7.2)} \qquad u_t - \beta(u)_{xx} = f \quad \text{for } 0 < t < T,\ \ell(t) > 0,\ 0 < x < \ell(t),$$

subject to

(7.3) $\ell(0) = \ell_o$ and $u(0, x) = u_o(x)$ for $0 < x < \ell_o$,

(7.4) $\beta(u)(t, 0) = g_o(t)$ for $0 < t < T$, $\beta(u)(t, \ell(t)) = 0$ for $0 < t < T$, and

$$\frac{d\ell(t)}{dt} = -\beta(u)_x(t, \ell(t)-) \qquad \text{for } 0 < t < T, \tag{7.5}$$

where $\ell_0 > 0$ is given as well as $u_0 \geqq 0$ in $W^{1,2}(0, \infty)$ with $u_0 = 0$ on $[\ell_0, \infty)$, $f \geqq 0$ in $L^2(0, T; H)$ and $g_0 \geqq 0$ in $W^{1,2}(0, T)$. By means of the family $\{\tilde{\phi}^t_{\ell,g_0};\ 0 \leqq t \leqq T\}$ of convex functions on H given by

$$\tilde{\phi}^t_{\ell,g_0}(z) = \begin{cases} \frac{1}{2}\int_0^\infty |z_x|^2 dx & \text{if } z \in K_{\ell,g_0}(t), \\ \infty & \text{otherwise} \end{cases}$$

with

$$K_{\ell,g_0}(t) = \{z \in W^{1,2}(0, \infty);\ z(0) = g_0(t),\ z(x) = 0 \text{ for } \ell(t) \leqq x < \infty\},$$

we can similarly give a quasi-variational formulation associated with system $\{(7.2) - (7.5)\}$, in which (7.5) is transformed into the integral equation

$$\ell(t)^2 = \ell_0^2 + 2\int_0^t g_0(r)dr + 2\int_0^{\ell_0} xu_0(x)dx - 2\int_0^\infty xu(t, x)dx$$
$$+ 2\int_0^t\int_0^{\ell(r)} xf(r, x)dxdr, \qquad \forall\ t \in [0, T].$$

Also in this case we can show the existence and uniqueness (in case $f \equiv 0$) of a solution to this quasi-variational problem by modifying the arguments developed in sections 2, 3, 4, 5 and 6.

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